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Global characteristics of jet impact

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This paper is concerned with the impact of a jet of arbitrary cross-section onto a rigid plane. At the initial stage of the impact the liquid motion is described within the framework of the acoustic approximation. Behind the shock front which is generated under the impact, the pressure distribution is calculated analytically for an arbitrary cross-section of the jet. It is shown that a quarter of the total impact energy transfers into the internal energy of the compressed liquid. The focusing of the compression and relief waves in the axisymmetrical case is discussed.

1. Introduction

This paper deals with the problem of the unsteady liquid flow caused by an impact on its boundary. Initially the liquid is at rest and occupies a cylindrical, half-infinite domain of arbitrary cross-section Ω. The side surface of the cylinder corresponds to the undisturbed position of the free liquid boundary, $(x, y) \in \partial\Omega, z > 0$. Below, the liquid region is bounded by a rigid undeformable plate $z = 0$ (figure 1 *a*). At some moment of time, which is taken as the initial one ($t = 0$), the rigid plate starts to move up at a constant velocity V. A shock wave and expansion waves are generated at the impact moment (figure 1 *b*).

We shall determine the pressure distribution $p(x, y, z, t)$ behind the shock front, the hydrodynamic force on the plane $F(t)$, the total impulse I, and the internal energy $E_i(t)$ of the disturbed liquid under the following assumptions: (i) the plate is solid and undeformable; (ii) the liquid is ideal and compressible; (iii) the Mach number $M = V/c_0$, where c_0 is the sound velocity in the liquid at rest, is much less than unity; and (iv) external mass forces and surface tension are absent.

This problem is of interest for some jet technologies, jet-printer performance, and also for cavitation erosion. It is known that the collapse of a bubble adjacent to a rigid surface leads to the formation of a microjet which hits the surface at a high velocity. The jet impingement on the surface is one of the causes of cavitation damage.

At the initial stage when the shock wave generated by the impact is not far from the plate, the compressibility of the liquid is of major significance. The liquid flow is described within the framework of the acoustic approximation provided that $M \ll 1$. This approximation is valid for impact speeds well below the medium's acoustic velocity and for times in which the deformations are small compared to the overall characteristic length. More details on the justification of the acoustic approach are given in Appendix A.

Dimensionless variables are used below. They are chosen so that both the sound velocity and the impact velocity are equal to unity, and the diameter of the jet cross-section is equal to 2. Then the scale of the pressure is the 'water hammer' pressure $\rho_0 c_0 V$, the scale of the liquid velocity is the impact velocity V, the length scale is half the diameter of the jet cross-section R, the time scale is the ratio R/c_0, ρ_0 is the density of the resting liquid.

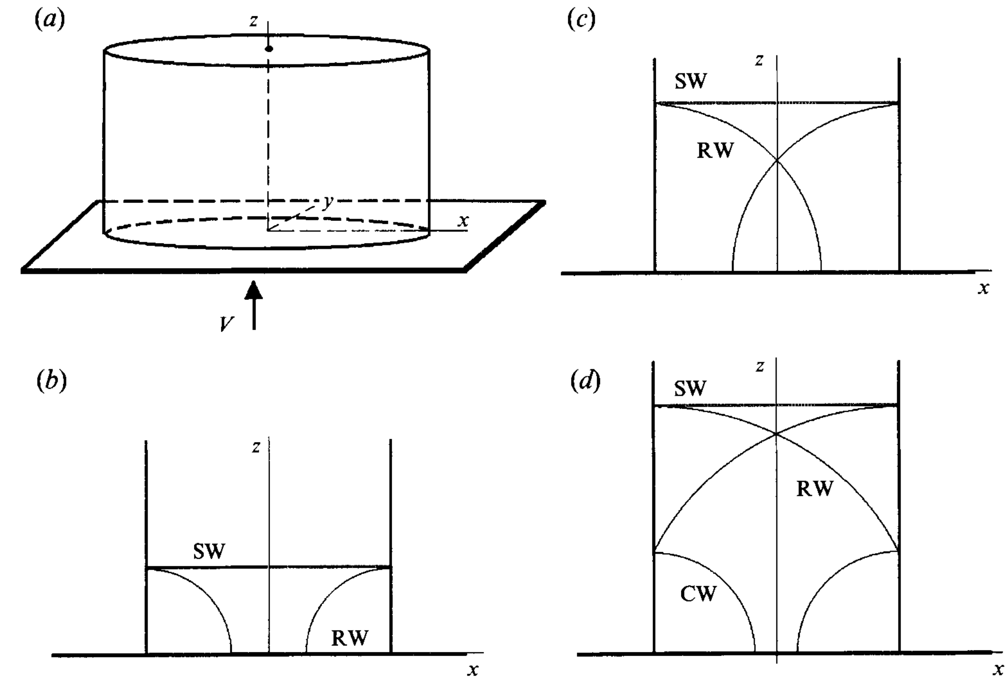

FIGURE 1. Impact of a plate on an half-infinite liquid cylinder. (*a*) Initially, liquid is at rest and occupies a cylinder $z > 0$ of arbitrary cross-section. The side boundary of the cylinder is free. (*b–d*) The wave pattern at the initial stage of the impact (plane case): (*b*) SW, shock wave; RW, relief wave produced at the impact moment. (*c*) Interaction of the relief waves. (*d*) CW, compression wave produced by the relief wave reflection in the free surface.

Both the plane and axisymmetrical problems are considered in detail. General formulae are presented for the impact of a jet of arbitrary cross-section. The present study is focused on the global characteristics of the process. In order to give a global description of the impact, a technique which does not account for details of the flow is used. This is a strength of the approach because the results of practical interest may be obtained and analysed in a quite simple way. On the other hand, this is also a weakness because the approach does not give much insight into the fluid mechanical aspects of the problem. For example, the pressure distribution inside a jet of arbitrary cross-section is found in this paper within the framework of the acoustic approximation. However, the corresponding formulae in the forms presented cannot be directly used for numerical calculations and further work is required. In particular, we cannot now compare the results given by the acoustic theory with many numerical simulations of the impact of a compressible liquid cylinder on a rigid boundary (Glenn 1974; Hwang & Hammitt 1977; Gonor & Yakovlev 1977; Pidsley 1982; Surov & Ageev 1989). A local description of the jet impact was given by Frankel (1990) and Veklich (1990) for the plane case. The axisymmetric case was analysed by Veklich (1991). For an arbitrary cross-section of the jet the present approach seems to be the only possible one. In order to make the formulae of the present paper available for numerical analysis, they should be improved by a well-known technique (see Kantorovich & Krylov 1962), the main idea of which is demonstrated in §4. The analysis of the flow and the pressure distribution inside the impacting jet, as well as the analysis of the validity of the acoustic theory applied to the jet impact problem is currently underway.

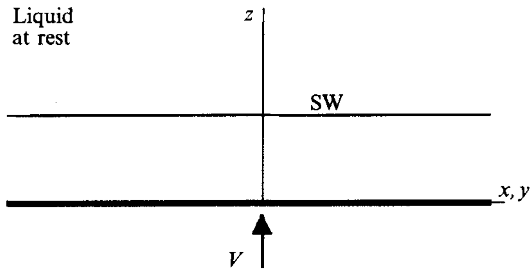

FIGURE 2. Creation of an acoustic shock, SW, by impact on a liquid half-space.

2. Formulation of the problem

Within the framework of the acoustic approximation the flow domain coincides with the cylinder $(x, y) \in \Omega, z \geqslant 0$ which is occupied by the liquid at the initial moment, $t = 0$. In non-dimensional variables the liquid flow is described by the velocity potential $\phi(x, y, z, t)$, for which the initial boundary-value problem has the form

$$\left.\begin{array}{l} \phi_{tt} = \phi_{xx} + \phi_{yy} + \phi_{zz} \quad ((x, y) \in \Omega, \quad z > 0), \\ \phi_z = 1 \quad ((x, y) \in \Omega, \quad z = 0), \\ \phi = 0 \quad ((x, y) \in \partial\Omega, \quad z > 0), \\ \phi = \phi_t = 0 \quad ((x, y) \in \Omega, \quad z \geqslant 0, \quad t < 0). \end{array}\right\} \tag{1}$$

Once the problem (1) has been solved, the normal derivative $\partial\phi/\partial \boldsymbol{n}$ on the free surface, which is equal to the normal velocity of the free surface displacement, can be calculated, and the position of this surface at moment t can be determined by integration in time. The velocity field is given by $\boldsymbol{u} = \nabla\phi$, $\boldsymbol{u} = (u, v, w)$, and the pressure by $p = -\phi_t$.

The original problem of jet impact contains the nonlinear equations of motion and boundary conditions which must be satisfied on the surface, the position of which is unknown in advance. Formally, the solution of (1) provides the asymptotics of the exact solution as $M \to 0$ under the assumption that the limiting values of the unknown functions and their first derivatives are finite (see Appendix A).

In order to analyse the pressure field inside the jet, let us first consider the problem without the free surface. This limiting problem corresponds to an impact on the boundary of the upper half-space $z > 0$ at a constant velocity (see figure 2). The solution of the limiting problem will be denoted by the subscript '0'. We obtain

$$\phi_0(x, y, z, t) = -(t-z)_+, \quad p_0(x, y, z, t) = H(t-z).$$

The notation $g_+ = g$ when $g > 0$, $g_+ = 0$ when $g \leqslant 0$ and $H(x) = 1$ when $x > 0$, $H(x) = 0$ when $x \leqslant 0$ are used. Problem (1) is linear, and therefore its solution can be presented in the form

$$\phi = \phi_0 + \varphi(x, y, z, t), \quad p = p_0 + q(x, y, z, t). \tag{2}$$

But now $\varphi_z = 0$ and, hence, $q_z = 0$ at $z = 0$, so that the functions $\varphi(x, y, z, t)$ and $q(x, y, z, t)$ can be continued in the region $(x, y) \in \Omega, z < 0$ symmetrically: $\varphi(x, y, -z, t) = \varphi(x, y, z, t)$ and $q(x, y, -z, t) = q(x, y, z, t)$. The boundary-value problem for the new unknown function $q(x, y, z, t)$ is formulated as

$$\left.\begin{array}{l} q_{tt} = q_{xx} + q_{yy} + q_{zz} \quad ((x, y) \in \Omega, \quad -\infty < z < +\infty), \\ q = -H(t^2 - z^2) \quad ((x, y) \in \partial\Omega, \quad -\infty < z < +\infty), \\ q = q_t = 0 \quad (t < 0). \end{array}\right\} \tag{3}$$

Problem (3) can be solved by the method of integral transforms with respect to the vertical coordinate z and the time t. Thereafter the pressure field will be given by (2). Solutions of (3) can be found in explicit forms for both the plane and axisymmetrical cases.

3. Plane problem

The plane problem (see figure 1*b–d*) was analysed by Frankel (1990) and Veklich (1990). They found both the pressure distribution and the velocity field inside the jet, as well as the deformation of the jet free surface. In the present paper a different method is used. This method can be applied to the problem of the impact by a jet of arbitrary cross-section.

The unknown function $q(x, y, z, t)$ is independent of y in the plane case. Problem (3) takes now the form

$$\left.\begin{aligned} &q_{tt} = q_{xx} + q_{zz} \quad (|x| < 1, \quad -\infty < z < +\infty), \\ &q = -H(t^2 - z^2) \quad (|x| = 1, \quad -\infty < z < +\infty), \\ &q = q_t = 0 \quad (t < 0). \end{aligned}\right\} \tag{4}$$

The Fourier transform with respect to the vertical coordinate z

$$q^F(x, \zeta, t) = \int_{-\infty}^{\infty} q(x, z, t)\,\mathrm{e}^{-\mathrm{i}z\zeta}\,\mathrm{d}z, \quad q(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q^F(x, \zeta, t)\,\mathrm{e}^{\mathrm{i}z\zeta}\,\mathrm{d}\zeta \tag{5}$$

and the Laplace transform with respect to time t

$$q^L(x, z, s) = \int_0^{\infty} q(x, z, t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \quad \operatorname{Re} s > 0,$$

$$q(x, z, t) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} q^L(x, z, s)\,\mathrm{e}^{st}\,\mathrm{d}s, \quad a > 0$$

applied to (4) yield

$$(s^2 + \zeta^2)\, q^{LF} = q^{LF}_{xx} \quad (|x| < 1),$$

$$q^{LF} = -\frac{2}{\zeta^2 + s^2} \quad (|x| = 1).$$

Therefore

$$q^{LF}(x, \zeta, s) = -\frac{2}{\zeta^2 + s^2}\,\frac{\cosh[(s^2 + \zeta^2)^{1/2}x]}{\cosh[(s^2 + \zeta^2)^{1/2}]}. \tag{6}$$

The solution $q^{LF}(x, \zeta, s)$ is an analytic function of the combination $s^2 + \zeta^2$ and has simple poles on the plane of s at the points for which $s^2 + \zeta^2 = 0$ and $(s^2 + \zeta^2)^{1/2} = \mathrm{i}(\frac{1}{2}\pi + \pi k)$ where $k = 0, \pm 1, \pm 2, \ldots$. Hence, in order to invert the Laplace transform, it is necessary to calculate the residues of $q^{LF}(x, \zeta, s)$ at these poles. The inversion of the Fourier transform leads to standard integrals and, taking (2) into account, we obtain the pressure distribution within the plane jet $z > 0, |x| < 1$:

$$\left.\begin{aligned} &p(x, z, t) = \frac{4}{\pi} S[\tfrac{1}{2}\pi x, \tfrac{1}{2}\pi(t^2 - z^2)^{1/2}]\, H(t - z), \\ &S(\alpha, \beta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[(2k+1)\alpha]\, J_0[(2k+1)\beta]. \end{aligned}\right\} \tag{7}$$

Notice that the pressure between the plate, $z = 0$, and the shock front, $z = t$, generated by the impact does not depend on the variables z, t separately but on their combination $(t^2-z^2)^{1/2}$ (see Veklich 1990). This means that the pressure profile $p(x, 0, t_1)$ travels along the jet and it is at the distance $z = (t^2-t_1^2)^{1/2}$ from the jet top at the instant $t, t > t_1$. The formula (7) is very convenient for evaluating global characteristics of the jet impact, as well as analysing the peculiarities of the pressure distribution.

In dimensionless variables the hydrodynamic force on the plate $F(t)$ is

$$F(t) = \int_{-1}^{1} p(x,0,t)\,\mathrm{d}x = \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{J_0[(k+0.5)\pi t]}{(2k+1)^2}. \tag{8}$$

Let us substitute the integral representation of the Bessel function

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos\theta)\,\mathrm{d}\theta$$

in (8) and change the order of the integration with respect to θ and the summation with respect to k. Then we obtain the Fourier series which can be evaluated analytically. The result is

$$F(t) = \frac{4}{\pi}\sum_{n=0}^{N}[(1+2\pi n)(\theta_n-\theta_{n+1})-t(\sin\theta_n-\sin\theta_{n+1})] \tag{9}$$

where $\theta_n = \arccos(2n/t), \quad \theta_{N+1} = 0, \quad N = [\tfrac{1}{2}t].$

Formula (9) presents the hydrodynamic force in the spirit of Frankel. Moreover, the same procedure applied to (7) gives the representation for the pressure distribution found by Frankel (1990) and Veklich (1990). In particular,

$$F(t) = 2-\frac{4}{\pi}t \quad (0 < t < 2),$$

$$F(t) = 2-\frac{4}{\pi}t-8\frac{\pi+1}{\pi}\arccos\frac{2}{t}+\frac{8}{\pi}(t^2-4)^{1/2} \quad (2 < t < 4).$$

The sum in (8) was numerically evaluated: 1000 terms were taken in the sum to guarantee accuracy of the result not lower than 10^{-4}. A graph of the force (8), figure 3, shows that a jet impact is not connected with large hydrodynamic loads only. The reason is that not only compression waves but also relief waves are formed under the impact. The latter result from the presence of the free surface of the jet and they move from the periphery of the contact region to its centre. The pressure drops in the region of the interaction of the relief waves, which leads to negative values of the hydrodynamic force $F(t)$. If the connection forces between the liquid particles and the rigid boundary are small enough, this phenomenon may lead to the separation of the liquid from the impacted surface with the appearance of cavities attached to the surface. Then the pressure increases again, which leads to a collapse of the bubbles with the formation of new compression waves. This suggests that the jet impact is not a simple phenomenon and that it may be governed by factors disregarded by classical hydrodynamics. The complicated character of the force evolution can be utilized to explain why the damage produced by the impact of a liquid drop or a pulsating jet may be much greater than that due to steady jet impact. Equation (8) shows that $F(t) = O(t^{-1/2})$ as $t \to \infty$. This means that the influence of acoustic effects on the flow decays quite slowly with time.

The work done by the hydrodynamic force is equal to the energy lost by the plate.

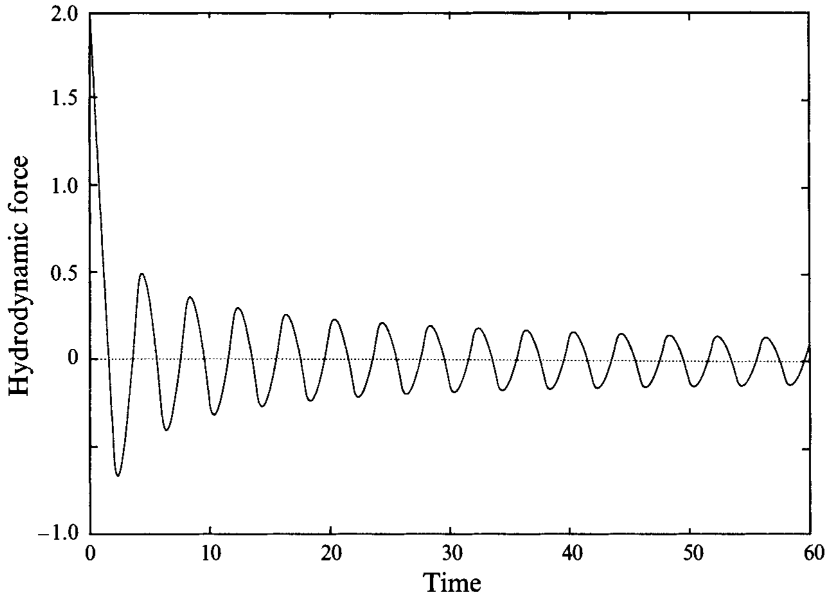

FIGURE 3. Hydrodynamic force $F(t)$ on a plate impacted by a plane jet.

The energy to be transferred to the plate to support the constant value of its velocity will be referred to as the external energy, $E_e(t)$. The work done during a small time interval Δt is equal to $F(t)\,\Delta t$ in dimensionless variables. Therefore

$$E_e(t) = \int_0^t F(\tau)\,\mathrm{d}\tau. \tag{10}$$

The limiting value of $E_e(t)$ as $t \to \infty$ is referred to as the total impulse, I. Taking into account (8) and the standard integral

$$\int_0^\infty J_0(cx)\,\mathrm{d}x = \frac{1}{c}, \quad c > 0, \tag{11}$$

we get

$$I = \frac{32}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{28}{\pi^3}\zeta(3) \tag{12}$$

where $\zeta(x)$ is the Riemann Zeta function. This quantity agrees with that given by Frankel (1990). In dimensionless variables the internal energy of compressed liquid $E_i(t)$ is

$$E_i(t) = \frac{1}{2}\int_{-1}^{1}\int_0^t p^2(x,z,t)\,dz\,\mathrm{d}x, \tag{13}$$

the scale of the energy being $\rho V^2 R^2$. Inserting (7) in (13), we obtain

$$E_i(t) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\int_0^t J_0^2[(k+0.5)\pi(t^2-z^2)^{1/2}]\,\mathrm{d}z.$$

Using the equality (see Appendix B)

$$\int_0^t J_0^2[c(t^2-z^2)^{1/2}]\,\mathrm{d}z = \frac{1}{2c}\int_0^{2tc} J_0(\tau)\,\mathrm{d}\tau, \tag{14}$$

we get the final formula

$$E_i(t) = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \int_0^{(2k+1)\pi t} J_0(\tau)\,\mathrm{d}\tau. \tag{15}$$

Taking the limit as $t \to \infty$ and taking into account (11) and (12), we can write that

$$E_i(\infty) = \tfrac{1}{4} E_e(\infty). \tag{16}$$

This shows that a quarter of the external energy is transferred to the internal energy of the compressed liquid, however small the impact velocity may be. This result is non-trivial because usually one connects a low-speed impact with an incompressible liquid model only, wherein this part of the energy is referred to as 'lost energy'.

The pressure distribution inside the plane jet can be analysed using the representation (7). However, the convergence of the series in (7) is quite weak. It can be improved by extracting the main part $S_m(\alpha, \beta)$ of the series $S(\alpha, \beta)$, but it is better to use the approach suggested by Frankel (1990) and Veklich (1990). Nevertheless, the function

$$S_m(\alpha, \beta) = \left(\frac{2}{\pi\beta}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{3/2}} \cos[\alpha(2k+1)] \cos[(2k+1)\beta - \tfrac{1}{4}\pi]$$

can be useful to study the asymptotic behaviour of the pressure as $\beta \to \infty$, based on the estimate

$$S(\alpha, \beta) = S_m(\alpha, \beta) + O(\beta^{-3/2})$$

which is valid as $\beta \to \infty$.

The function $S_m(\alpha, \beta)$ can be presented in quadratures. Analysis of the integral representation demonstrates that the pressure $p(x, z, t)$ is finite everywhere inside the jet, and the pressure gradient is unbounded near the curves $z^2 + (1 + 2N \pm x^2)^2 = t^2$ where $N = 0, 1, 2, \ldots$. In particular, narrow zones of high pressure gradients appear periodically at the centre of the contact region at the time instants $t_N = 1 + 2N$. For more details of the pressure distribution see Frankel (1990) and Veklich (1990).

4. Axisymmetric problem

Axisymmetric jet impact was first analysed by Veklich (1991). Here the approach suggested in the previous section is used. In the polar coordinate system r, θ, $r = (x^2 + y^2)^{1/2}$ (figure 4), the function $q(x, y, z, t)$ is independent of θ and satisfies

$$\left.\begin{aligned} q_{tt} &= q_{rr} + \frac{1}{r} q_r + q_{zz} \quad (r < 1, \quad -\infty < z < +\infty), \\ q &= -H(t^2 - z^2) \quad (r = 1, \quad -\infty < z < +\infty), \\ q &= q_t = 0 \quad (t < 0). \end{aligned}\right\} \tag{17}$$

The Fourier transform with respect to z and the Laplace transform with respect to time t, applied to (17), yield

$$(s^2 + \zeta^2)\, q^{LF} = q^{LF}_{rr} + \frac{1}{r} q^{LF}_r \quad (r < 1),$$

$$q^{LF} = -\frac{2}{\zeta^2 + s^2} \quad (r = 1).$$

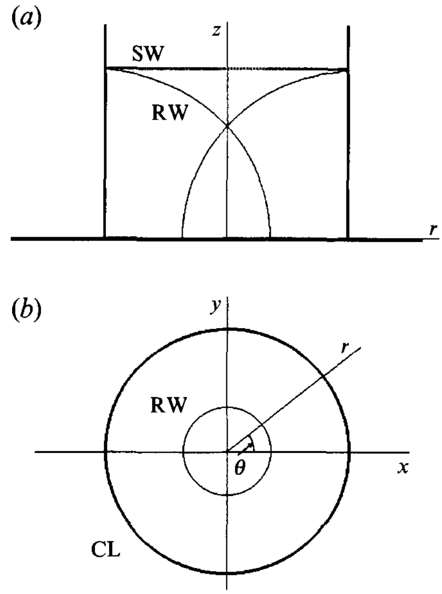

FIGURE 4. Impact of a plate on a liquid circular cylinder. (*a*) The side view of the wave pattern at the initial stage of the impact (axisymmetric case): SW, shock wave; RW, relief wave; *r*, radial coordinate. (*b*) The bottom view: CL, contact line between the liquid free surface and the plate; relief wave, RW, converges to the centre point.

The solution of this boundary-value problem, bounded at the jet centre, $r = 0$, is

$$q^{LF}(r,\zeta,s) = -\frac{2}{\zeta^2+s^2}\frac{I_0[r(s^2+\zeta^2)^{1/2}]}{I_0[(s^2+\zeta^2)^{1/2}]}, \tag{18}$$

where $I_0(r)$ is the modified Bessel function of zero order. It is seen that $q^{LF}(x,\zeta,s)$ is the analytic function of the combination $s^2+\zeta^2$ and has simple poles on the s-plane at the points where $s^2+\zeta^2 = 0$ and $(s^2+\zeta^2)^{1/2} = \mathrm{i}\mu_k$, $k = 1,2,\ldots,\mu_k$ being the roots of the equation $J_0(\mu) = 0$ and $\mu_1 < \mu_2 < \ldots < \mu_n < \ldots$. That is why we need only to determine the residues of $q^{LF}(r,\zeta,s)$ at these poles, to inverse the Laplace transform. We obtain

$$p(r,z,t) = 2\sum_{k=1}^{\infty}\frac{J_0(r\mu_k)\,J_0[\mu_k(t^2-z^2)^{1/2}]}{\mu_k J_1(\mu_k)}H(t-z) \tag{19}$$

where $z > 0$, $0 < r < 1$, $J_0(r)$, $J_1(r)$ are the Bessel functions of the zero and first orders, respectively. It is seen that in the axisymmetrical case the pressure also depends on the combination $(t^2-z^2)^{1/2}$. This means that to find the pressure distribution inside the jet, one needs to determine the pressure $p(r,0,t)$ on the plate only. Formula (19) was derived by Veklich (1991) by a different method.

In dimensionless variables the hydrodynamic force $F(t)$ on the plate is equal to

$$F(t) = 2\pi\int_0^1 rp(r,z,t)\,\mathrm{d}r = 4\pi\sum_{k=1}^{\infty}\frac{J_0(\mu_k t)}{(\mu_k)^2}. \tag{20}$$

In order to derive (20), the standard integral

$$\int_0^1 rJ_0(r\mu_k)\,\mathrm{d}r = \frac{1}{\mu_k}J_1(\mu_k)$$

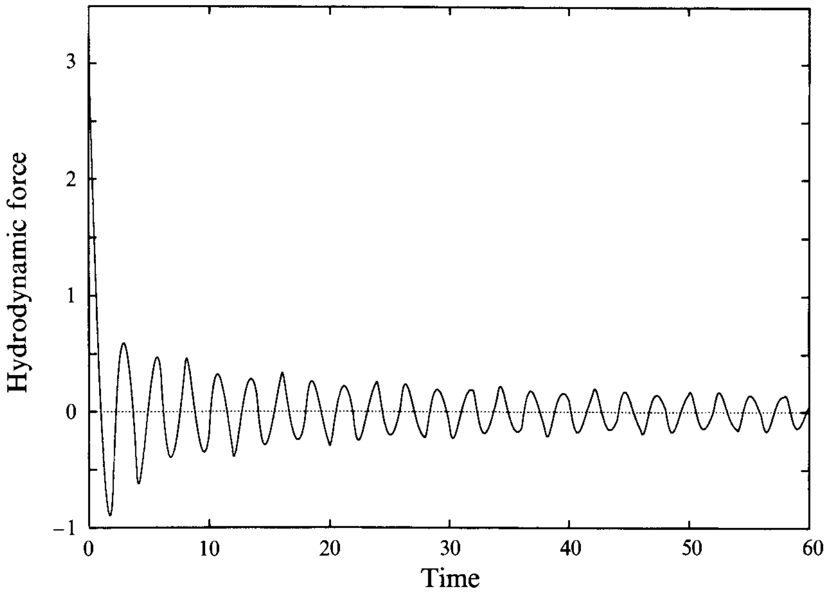

FIGURE 5. Hydrodynamic force $F(t)$ on a plate impacted by an axisymmetrical jet.

was used. The graph of the force (20) is shown in figure 5. The main features of the force evolution shown for the plane jet impact are valid in the axisymmetrical case as well. However, now the change of the force with time is more abrupt. This is connected with the fact that the interaction of the torus-like rarefaction waves and that of the torus-like compression waves formed under an axisymmetric jet impact yield a greater amplitude of pressure variation than in the plane-jet impact.

The total impulse I for the axisymmetrical jet impact is

$$I = 4\pi \sum_{k=1}^{\infty} \frac{1}{(\mu_k)^2} \int_0^{\infty} J_0(\mu_k t)\,\mathrm{d}t = 4\pi \sum_{k=1}^{\infty} \frac{1}{(\mu_k)^3}$$

where (11) was used. Numerical calculations give $I = 1.016355$.

The non-dimensional internal energy $E_i(t)$ is given in this case by

$$E_i(t) = \pi \int_0^1 \int_0^t r p^2(r, z, t)\,\mathrm{d}z\,\mathrm{d}r. \tag{21}$$

Inserting (19) into (21) and taking into account (14) and

$$\int_0^1 r J_0(r\mu_k)\, J_0(r\mu_n)\,\mathrm{d}r = \tfrac{1}{2} J_1^2(\mu_k)\,\delta_{k,n}$$

(Gradshteyn & Ryzhik 1980), we find

$$E_i(t) = \pi \sum_{k=1}^{\infty} \frac{1}{\mu_k^3} \int_0^{2t\mu_k} J_0(\tau)\,\mathrm{d}\tau. \tag{22}$$

This means that relation (16) is also valid in the axisymmetrical case.

However, the pressure distribution is more complicated than that in the plane problem. In the axisymmetrical case both compression and rarefaction waves are torus-like; therefore when they interact extremely low or high values of the pressure may be expected. Let us consider the pressure evolution at the centre of the contact region, $r = 0$, $z = 0$, $t > 0$:

$$p(0,0,t) = \sum_{k=1}^{\infty} D_k(t), \quad D_k(t) = 2\frac{J_0(\mu_k t)}{\mu_k J_1(\mu_k)}.$$

This series may converge only conventionally because $D_k(t) = O(k^{-1})$ as $k \to \infty$. Moreover, it is possible that $p(0,0,t)$ is unbounded at some instants of time. In order to clarify the characteristics of the pressure at the centre point, we need to determine the asymptotic behaviour of the summands $D_k(t)$ as $k \to \infty$. Taking into account

$$\mu_k = (k - \tfrac{1}{4})\pi + O(k^{-1}), \quad J_0(\mu_k t) = \frac{1}{(\mu_k t)^{1/2}} \cos(\mu_k t - \tfrac{1}{4}\pi) + O(k^{-3/2}),$$

$$J_1(\mu_k) = \frac{1}{\mu_k^{1/2}} \cos(\mu_k - \tfrac{3}{4}\pi) + O(k^{-3/2}),$$

which are uniformly valid where $k \gg 1$, $t > 0$, we obtain the asymptotic formula

$$D_k(t) = D_k^{(0)}(t) + D_k^{(1)}(t),$$

$$D_k^{(0)}(t) = \frac{2}{\pi t^{1/2}} \frac{(-1)^{k-1}}{k} \cos[k\pi t - \tfrac{1}{4}\pi(t+1)], \quad D_k^{(1)}(t) = O(k^{-2}).$$

Therefore, $p(0,0,t) = p^{(0)}(t) + p^{(1)}(t)$ where

$$p^{(0)}(t) = \sum_{k=1}^{\infty} D_k^{(0)}(t), \quad p^{(1)}(t) = \sum_{k=1}^{\infty} D_k^{(1)}(t).$$

The second sum converges absolutely, hence $p^{(1)}(t)$ is a bounded function and it can be evaluated numerically. The first sum converges conventionally, but it can be evaluated analytically. This simple idea makes it possible not only to analyse the characteristics of the pressure distribution, but also to evaluate the pressure numerically in an effective way. We obtain

$$p(0,0,t) = \frac{2}{\pi t^{1/2}} [\ln[2\cos(\tfrac{1}{2}\pi(t-2N))] \cos(\tfrac{1}{4}\pi(t+1)) + \tfrac{1}{2}\pi(t-2N)\sin(\tfrac{1}{4}\pi(t+1))] + p^{(1)}(t),$$

where the integer N is such that $|t - 2N| < 1$. Analysis of the wave front kinematics shows that expansion waves converge to the jet centre at the instants of time $t = 4n+1$, $n = 0, 1, \ldots$ and compression waves converge at $t = 4n+3$. Calculating $p^{(0)}(t)$ at these instants, we conclude that while the expansion waves are interacting the pressure can be quite low but finite, whereas the interaction of the compression waves leads to the logarithmic growth of the pressure amplitude.

The liquid flow and the pressure distribution inside the circular jet were numerically determined by Veklich (1991) with the help of (19). Unfortunately, Veklich does not report any details of the numerical procedure. In particular, the accuracy of the numerical results is not clear. The numerical results show strange behaviour of the relief and compression waves inside the circular jet. For example, a relief wave can reflect from the jet axis as either a relief wave or a compression one. Physically, this phenomenon is obscure. We expect that the analysis of the dynamics of waves inside a circular jet by the geometric acoustic approach in the spirit of Lesser (1981) will be able to clarify this point.

5. General case

For an arbitrary region Ω (figure 6), we can introduce the eigenfunctions $A_k(x,y)$ and the eigenvalues λ_k, which satisfy the equations

$$\frac{\partial^2 A_k}{\partial x^2}+\frac{\partial^2 A_k}{\partial y^2}+\lambda_k^2 A_k = 0 \quad ((x,y)\in\Omega),$$

$$A_k = 0 \quad ((x,y)\in\partial\Omega)$$

and the orthogonality condition

$$\int_\Omega A_k(x,y)\,A_n(x,y)\,\mathrm{d}x\,\mathrm{d}y = \delta_{n,k}, \tag{23}$$

where $\delta_{n,k} = 1$ when $n = k$ and $\delta_{n,k} = 0$ when $n \neq k$. Generally speaking, there can be some eigenfunctions $A_{kj}(x,y), j = 1,2,\ldots,N_k$ which correspond to the same eigenvalue λ_k. For simplicity, we shall consider only the case $N_k \equiv 1$.

Let us denote

$$c_k = \int_\Omega A_k(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

then the pressure distribution in the jet can be proved to be given by

$$p(x,y,z,t) = \sum_{k=0}^{\infty} c_k\,A_k(x,y)\,J_0[\lambda_k(t^2-z^2)^{1/2}]\,H(t-z). \tag{24}$$

It is easy to see that this function satisfies the wave equation and the boundary condition on the free surface of the jet. Therefore, we need to prove only that

$$\lim_{z\to+0} p_z(x,y,z,t) = -\delta(t) \quad ((x,y)\in\Omega),$$

where $\delta(t)$ is the Dirac delta function. Inserting (24) in this condition, then multiplying by $A_n(x,y)$ and integrating over Ω both its sides, and using (23), we find

$$c_n \lim_{z\to+0} \frac{\mathrm{d}}{\mathrm{d}z}[J_0[\lambda_n(t^2-z^2)^{1/2}]\,H(t-z)] = -\delta(t)\,c_n.$$

Here

$$\frac{\mathrm{d}}{\mathrm{d}z}[J_0[\lambda_n(t^2-z^2)^{1/2}]\,H(t-z)]$$

$$= \lambda_n\frac{z}{(t^2-z^2)^{1/2}}J_1[\lambda_n(t^2-z^2)^{1/2}]\,H(t-z)-J_0[\lambda_n(t^2-z^2)^{1/2}]\,\delta(t-z)$$

and one can see that the first term vanishes as $z\to+0$ and the second one tends to $-\delta(t)$. This reasoning proves the representation (24).

In the non-dimensional variables the hydrodynamic force on the plate $F(t)$ is

$$F(t) = \int_\Omega p(x,y,0,t)\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=0}^{\infty} c_k^2\,J_0(\lambda_k\,t)$$

and the total impulse I is given by

$$I = \int_0^\infty F(t)\,\mathrm{d}t = \sum_{k=0}^{\infty}\frac{c_k^2}{\lambda_k}.$$

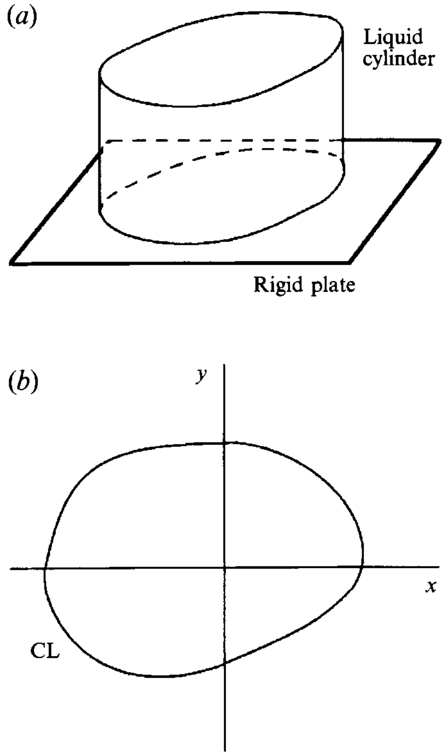

FIGURE 6. Impact of a plate on a liquid cylinder of arbitrary cross-section. (*a*) General view at the initial moment. (*b*) The bottom view: CL, contact line between the liquid free surface and the plate.

The internal energy of compressed liquid $E_i(t)$ is determined in non-dimensional variables by

$$E_i(t) = \frac{1}{2}\int_0^t \left[\iint_\Omega p^2(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}z. \tag{25}$$

Inserting (24) into (25) and taking into account (23) and (14), we obtain

$$E_i(t) = \frac{1}{4}\sum_{k=0}^{\infty}\frac{c_k^2}{\lambda_k}\int_0^{2t\lambda_k} J_0(\tau)\,\mathrm{d}\tau$$

and, hence, (16) is valid for a jet of arbitrary cross-section as well.

In the plane case, we have

$$A_k(x) = \cos((2k+1)\tfrac{1}{2}\pi x), \quad c_k = \frac{4}{\pi}\frac{(-1)^k}{2k+1}, \quad \lambda_k = \tfrac{1}{2}\pi(2k+1),$$

and (24) leads to (7). Accordingly, in the axisymmetrical case, we have

$$A_k(r) = \frac{J_0(r\mu_k)}{\pi^{1/2}J_1(\mu_k)}, \quad c_k = \frac{2\pi^{1/2}}{\mu_k}, \quad \lambda_k = \mu_k$$

and (24) leads to (19).

6. Jet impact onto a permeable surface

This problem is relevant to jet printer performance. In particular, the liquid mass which can penetrate a paper surface under jet impact can be of practical interest. The simplest model is considered here. Within the framework of this model the velocity of the liquid penetrating the surface is taken to be proportional to the local hydrodynamic pressure. The liquid flow is governed by equations (1) where the condition on the surface, $z = 0$, $(x,y)\in\Omega$, has to be replaced by

$$\phi_z = 1-\alpha p.$$

Here α is a positive coefficient dependent of the surface properties, $0<\alpha<1$.

The pressure will be sought in the form of the Fourier series

$$p(x,y,z,t) = \sum_{k=0}^{\infty} M_k(t,z)\,A_k(x,y) \tag{26}$$

where the coefficients $M_k(t,z)$ satisfy

$$\frac{\partial^2 M_k}{\partial t^2} = \frac{\partial^2 M_k}{\partial z^2}-\lambda_k^2 M_k \quad (z>0),$$

$$\frac{\partial M_k}{\partial z}-\alpha\frac{\partial M_k}{\partial t} = -\delta(t)\,c_k \quad (z=0),$$

$$M_k\to 0 \quad (z\to\infty).$$

Using the Laplace transform, one obtains

$$M_k^L(s,0) = \frac{c_k}{(s^2+\lambda_k^2)^{1/2}+\alpha s}$$

and

$$\frac{1}{c_k}M_k(t,0) = J_0(\lambda_k t)-\frac{\alpha}{1-\alpha^2}G\left(\frac{\lambda_k t}{(1-\alpha^2)^{1/2}},\alpha\right),$$

$$G(z,\alpha) = \cos z+\alpha\int_0^z J_0[(1-\alpha^2)^{1/2}v]\sin(z-v)\,\mathrm{d}v.$$

Notice that $G(z,\alpha)\to 0$ as $z\to\infty$ and $0\leqslant\alpha<1$. The asymptotics of $G(z,\alpha)$ as $\alpha\to 0$ is non-uniform; nevertheless we can write that

$$\frac{1}{c_k}M_k(t,0)-J_0(\lambda_k t) = O(\alpha).$$

The hydrodynamic force on the impacted plate is given by

$$F(t,\alpha) = \sum_{k=0}^{\infty} M_k(t,0)\,c_k$$

and it vanishes as $O(t^{-1/2})$ for $t\to\infty$. It is observed that

$$F(0,\alpha) = \left(1-\frac{\alpha}{1-\alpha^2}\right)F(0,0).$$

Thus, $F(0,\alpha)>0$ for $0<\alpha<\frac{1}{2}(\sqrt{5}-1)$. In the time interval $0\leqslant t\leqslant 2$, we find in the plane case that

$$F(t,\alpha) = 2\left(1-\frac{\alpha}{1-\alpha^2}\right)-\frac{4t}{\pi}\left(1-\frac{\alpha}{(1-\alpha^2)^{3/2}}\arctan\frac{(1-\alpha^2)^{1/2}}{\alpha}\right).$$

Physically, it follows that the force decreases in time at the initial stage. This leads to the restriction $0 \leqslant \alpha \leqslant \alpha_1$ where α_1 is the solution of

$$\arctan\frac{(1-\alpha_1^2)^{1/2}}{\alpha_1} = \frac{(1-\alpha_1^2)^{3/2}}{\alpha_1}.$$

The mass of liquid $m(t, \alpha)$ which has penetrated the surface up to time t is determined in non-dimensional variables by

$$m(t,\alpha) = \int_0^t \left(\int_\Omega \alpha p(x,y,0,\tau)\, \mathrm{d}x\, \mathrm{d}y \right) \mathrm{d}\tau = \alpha \int_0^t F(\tau,\alpha)\, \mathrm{d}\tau.$$

The integrals

$$\int_0^\infty M_k(t,0)\, \mathrm{d}t$$

exist and, therefore,

$$\int_0^\infty M_k(t,0)\, \mathrm{d}t = \lim_{s\to 0} \frac{c_k}{(s^2+\lambda_k^2)^{1/2}+\alpha s} = \frac{c_k}{\lambda_k}.$$

This means that

$$m(\infty,\alpha) = \alpha I(\infty,0), \tag{27}$$

i.e. the total liquid mass which will be absorbed in the permeable body after the jet impact is proportional to the total impulse determined for a rigid surface.

7. Impact of a finite liquid cylinder

For a finite liquid cylinder (figure 7) of length H in non-dimensional variables, representation (26) remains valid. The coefficients $M_k(t,z)$ now satisfy the equations

$$\frac{\partial^2 M_k}{\partial t^2} = \frac{\partial^2 M_k}{\partial z^2} - \lambda_k^2 M_k \quad (0 < z < H),$$

$$M_k = 0 \quad (z = H),$$

$$\frac{\partial M_k}{\partial z} = -\delta(t)\, c_k \quad (z = 0).$$

Using the Laplace transform, we find

$$M_k^L(s,z) = \frac{\sinh[(s^2+\lambda_k^2)^{1/2}(H-z)]}{(s^2+\lambda_k^2)^{1/2}\cosh[(s^2+\lambda_k^2)^{1/2}H]}\, c_k.$$

In particular,

$$\int_0^\infty M_k(\tau,0)\, \mathrm{d}\tau = \frac{\tanh\lambda_k H}{\lambda_k}\, c_k,$$

and the total impulse I for the finite jet is

$$I(H) = \sum_{k=0}^\infty c_k^2 \frac{\tanh\lambda_k H}{\lambda_k}.$$

It is clear that $I(H) < I(\infty)$ and

$$\frac{I(\infty)-I(H)}{I(\infty)} < 1-\tanh\lambda_0 H,$$

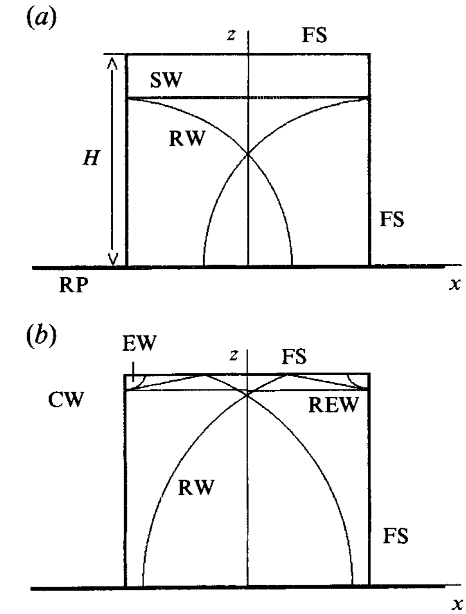

FIGURE 7. Impact by a finite liquid cylinder (a plane case is shown). (*a*) The wave pattern at the initial stage when the presence of the upper free surface does not affect the flow: FS, free surface; SW, shock wave; RW, relief wave; RP, rigid plate; *H*, length of the cylinder. (*b*) Reflection of the shock wave at the upper free surface of the cylinder: REW, reflected expansion wave; EW, edge wave; CW, compression wave.

where λ_0 is the minimal eigenvalue. Therefore, in the case

$$H \geqslant \lambda_0^{-1} \ln\left(\frac{2}{\epsilon}-1\right)^{1/2}$$

one can take $I(H) \approx I(\infty)$ with relative error ϵ. For $\epsilon = 0.01$ we find $H \geqslant 1.7$ in the plane case and $H \geqslant 1.1$ in the axisymmetrical one.

This problem has been the subject of intensive numerical studies (see Glenn 1974; Huang, Hammitt & Yang 1973; Hwang & Hammitt 1977; Gonor & Yakovlev 1977; Surov & Ageev 1989) in connection with the erosion of rigid surfaces by drop impact. The erosion is a complex phenomenon which essentially depends on the geometries of both the liquid drop and the impacted boundary. Impact by a circular liquid cylinder onto a undeformable plate is a particular case of the general problem, which is quite suitable for numerical analysis. Nevertheless, the numerical simulation of the liquid cylinder impact is non-trivial, mainly owing to the complex pattern of the flow. The numerical results agree very well with experimental data but imperfectly with the theoretical predictions (see, for example, the discussion by F. J. Heymann following the paper by Huang *et al.* 1973). The careful analysis of the problem within the framework of the acoustic approximation is expected to give the information on the characteristics of the flow which will be helpful for construction of adequate numerical schemes.

Impact by a finite liquid cylinder in both the plane and axisymmetric cases was studied by Veklich (1990, 1991) within the framework of the acoustic approximation. Numerical results were presented for the hydrodynamic force on the impacted rigid surface. Attention was focused on the compression of a liquid drop between two parallel plates.

8. Conclusion

It is shown in this paper that the pressure distribution inside a jet which hits a rigid plate can be found in the form of Fourier series and has a special structure. Convergence of the series is not good enough for direct numerical calculations, and so a careful analysis of the series is required. On the other hand, global characteristics (hydrodynamic force, total impulse, internal energy) can be obtained quite easily for arbitrary cross-section of the jet. It was shown that a quarter of the impact energy is transferred into the internal energy of the compressed liquid. The pressure inside the jet, as well as the total hydrodynamic force, may be negative. This indicates the possibility of cavitation under the impact. Cavitation depends on the liquid properties, and the present case is ideal because it was assumed that the liquid cannot disintegrate or separate from the impacted surface (interface cavitation), however low the hydrodynamic pressure may be.

The permeability of the impacted surface affects the evolution of the jet impact. However, the total liquid mass that penetrates the surface may be obtained using (27) and is proportional to the total impulse determined for the rigid surface.

The dependence of the total impulse on the jet length may be disregarded if the length is greater than the diameter of the jet cross-section.

The acoustic approximation is quite rough for presenting a complete picture of the jet impact. Nevertheless, we expect that the values of global characteristics which are connected with average quantities but not with local ones are satisfactorily predicted by the acoustic approach.

The author would like to express his thanks to Professor V. M. Teshukov for helpful discussions. Preliminary results of this work were announced at the IUTAM Symposium on Bubble Dynamics and Interface Phenomena, Birmingham, England, September, 1993.

Appendix A. Justification of the acoustic approximation

The problem of plane jet impact is considered. It is not difficult to extend the analysis presented to the impact of a jet of an arbitrary cross-section. We shall describe general features of the flow inside the jet and justify the acoustic approximation for impact velocities which are well below the sound speed at the resting liquid. Dimensional variables are used below.

We assume that the fluid of the jet is non-viscous, perfect and does not conduct heat (see Timman 1960). This means that the equation of state is

$$P = \mathscr{R} T \rho, \tag{A 1}$$

the entropy S is defined by the differential relation

$$T \,\mathrm{d}S = \mathrm{d}e + P\,\mathrm{d}(1/\rho), \tag{A 2}$$

and the specific internal energy e by

$$e = c_v T. \tag{A 3}$$

Here P is the pressure, ρ is the density, T is the temperature, the constants $\mathscr{R}$ and c_v are dependent on the fluid properties. Equations (A 1)–(A 3) yield

$$P = A(S)\rho^{\gamma}, \tag{A 4}$$

$$e = P/[\rho(\gamma-1)], \tag{A 5}$$

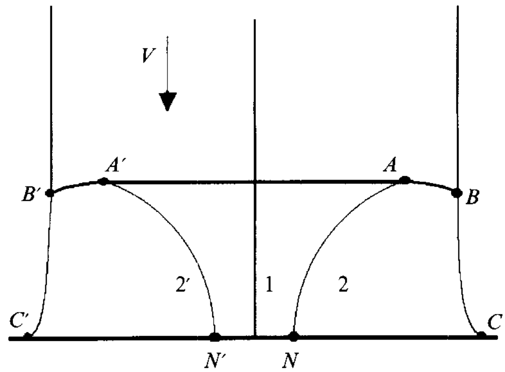

FIGURE 8. The wave pattern at the initial stage of the jet impact (nonlinear theory, plane case): AA', undeformable part of the shock front; AB and $A'B'$, curved parts of the shock; AN and $A'N'$, fronts of the relief waves; BC and $B'C'$, disturbed parts of the jet free surface; V, jet speed. The liquid is at rest in region $ANN'A'$, the flows inside regions $ABCN$ and $A'N'C'B'$ are self-similar at the stage shown.

where $A(S) = \mathscr{R}\exp[(S-S_*)/c_v]$, S_* is constant, $\gamma = 1+\mathscr{R}/c_v$, and the local sound velocity in the flow is given by

$$c = (\gamma P/\rho)^{1/2}. \tag{A 6}$$

The constant S_* can be taken without loss of generality to be zero.

Initially the liquid is at rest and occupies the region $|x| < L, z > 0$. The liquid density is ρ_0 and the sound velocity is c_0. The initial pressure P_0 is given by (A 6): $P_0 = \rho_0 c_0^2/\gamma$, the initial value of the specific internal energy e_0 by (A 5): $e_0 = c_0^2/[\gamma(\gamma-1)]$, the initial entropy S_0 by (A 4).

At $t = 0$ the liquid particles instantly obtain the velocity which is directed down to the rigid boundary and is equal to V. A sketch of the flow at the initial stage is shown in figure 8. The central part of the shock front AA' is plane, its dimension thus vanishes with time. Parts AB and $A'B'$ of the shock are curved owing to the interaction of the shock wave with the relief waves which move from the free surface of the jet, BC and $B'C'$, to its centre. At the stage under consideration the disturbed flow may be divided into the three regions: 1, 2, 2′. The flow inside region 1 is not affected by the presence of the liquid free surface and, therefore, is the same as that under the impact of the liquid half-plane. Region 1 is bounded below by the rigid wall, $z = 0$, where the normal velocity of the liquid is zero, and above by the plane part of the shock front AA', where

$$\rho_1 J = \rho_0(J+V), \tag{A 7}$$

$$P_1+\rho_1 J^2 = P_0+\rho_0(J+V)^2, \tag{A 8}$$

$$\frac{P_1}{P_0} = \frac{(\gamma+1)\rho_1-(\gamma-1)\rho_0}{(\gamma+1)\rho_0-(\gamma-1)\rho_1}; \tag{A 9}$$

quantities in this region are denoted by the subscript '1', J is the normal velocity of the shock front. Region 1 is also bounded by the fronts of the relief waves AN and $A'N'$. The liquid in region 1 is at rest: all hydrodynamic and thermodynamic quantities are constant here and can be found with the help of (A 4)–(A 9). They are

$$\rho_1 = \rho_0 R, \quad J = \frac{V}{R-1}, \quad \Delta := \frac{P_1-P_0}{\rho_0 c_0^2} = \frac{RM^2}{R-1},$$

$$c_1 = c_0\left(\frac{1+\gamma\Delta}{R}\right)^{1/2}, \quad e_1 = \frac{c_1^2}{\gamma(\gamma-1)},$$

where $M = V/c_0$ is the Mach number, R is the solution of the quadratic equation

$$[2+M^2(\gamma-1)]R^2-[4+M^2(\gamma+1)]R+2 = 0,$$

which tends to unity as $M \to 0$, and Δ is usually referred to as the strength of the shock. When the jet speed is much smaller than the sound velocity, $M \ll 1$, the following asymptotic relations can be found:

$$\left.\begin{aligned} R = 1+M-\tfrac{1}{4}(\gamma-3)M^2+O(M^3), \quad \Delta = M+\tfrac{1}{4}(\gamma+1)M^2+O(M^3),\\ J = c_0(1+\tfrac{1}{4}(\gamma-3)M+O(M^2)), \quad c_1 = c_0(1+\tfrac{1}{2}(\gamma-1)M+O(M^2)),\\ e_1 = e_0(1+(\gamma-1)M+O(M^2)), \quad S_1-S_0 = O(M^3). \end{aligned}\right\} \tag{A 10}$$

Shock waves with strength much less than unity (in our case, $\Delta = O(M)$) are referred to as weak shock waves. For an arbitrary weak shock, $\Delta \ll 1$, equations (A 4), (A 9) give

$$\frac{S_1-S_0}{c_v} = \frac{\gamma}{12}(\gamma^2-1)\Delta^3+O(\Delta^4). \tag{A 11}$$

The flow behind a shock cannot be isentropic, but for weak shocks the flow may be considered as approximately isentropic in view of the estimate (A 11). The value of Δ can be estimated as $O(M)$ because the strength of the shock on BA and $B'A'$ may be less than on AA' but not greater.

The fronts AN and $A'N'$ of the relief waves are lines of weak discontinuities of the flow parameters. This means that the pressure and the liquid velocity are continuous near these lines, while some of their derivatives are not. The normal velocity of these lines is equal to the local sound velocity which is c_1. The relief waves appear at the moment of impact on the periphery of the contact region. Hence, the line AN is a circular arc with radius $c_1 t$ and centre at the point $x = L$, $z = 0$, and the line $A'N'$ is a circular arc with the same radius and centre at the point $x = -L, z = 0$. The coordinates x_A, z_A of point A satisfy the equations

$$z_A = Jt, \quad (x_A-L)^2+z_A^2 = c_1^2 t^2$$

which predict

$$x_A = L-(c_1^2-J^2)^{1/2}\,t.$$

Points A and A' move to the centre of the jet at the constant speed $(c_1^2-J^2)^{1/2}$. Taking (A 10) into account, we obtain for the weak shock

$$(c_1^2-J^2)^{1/2} = c_0 M^{1/2}[(\gamma+1)/2]^{1/2}+O(c_0 M^{3/2}).$$

Thus, both the parameters of the liquid state in region 1 and the geometry of the region can be determined for arbitrary jet speed and arbitrary jet cross-section. Region 1 disappears at $t_1 = L/(c_1^2-J^2)^{1/2}$.

Within the framework of the acoustic approximation the velocities of the shock front and relief waves are equal to the sound velocity at the resting liquid c_0 that corresponds to the leading-order terms in (A 10). Therefore in this approximation the curved parts of the shock front, AB and $A'B'$, are absent. This approximation is unable to describe the appearance of the curved parts of the shock front and the fine structure of the flow near them. This is well-known; it indicates that the acoustic approximation is not valid near points where the shock touches the free surface. But the dimensions of these areas are small and tend to zero as $M \to 0$. Indeed, at the initial time interval considered in the acoustic approximation we have $tc_0/L = O(1)$, therefore the relative size of the curved shock part is $(L-x_A)/L = (tc_0/L)[(\gamma+1)/2]^{1/2}M^{1/2}+\ldots = O(M^{1/2})$ and vanishes as $M \to 0$. Thus the presence of points A and A' is connected with the nonlinear effects which are localized near points B and B'. The horizontal size of the areas is of $O(M^{1/2})$ and their vertical size is of $O(M)$ as $M \to 0$. Inside these areas the acoustic theory has to be improved, which can be achieved by asymptotic methods. This problem is not considered here.

In the plane problem the liquid flow and the pressure distribution in region 2 are self-similar and depend on the variables $(x-L)/(c_0 t), z/(c_0 t)$. This means, in particular, that the shape of the free surface BC is given by $x = L + Vt\zeta(z/c_0 t)$ where the function $\zeta = \zeta(\xi), \xi = z/c_0 t$ should be determined for $0 \leqslant \xi \leqslant v_B/c_0$, $v_B = \mathrm{d}z_B(t)/\mathrm{d}t$, $\zeta(v_B/c_0) = 0$. Therefore, there is no initial asymptotics of the solution in time in contrast to the impact problem studied by King & Needham (1994).

At $t_2 = L/c_1$ the relief waves (AN and $A'N'$) reach the jet centre and their interaction starts. This interaction leads to a pressure drop and to a decrease of the speed of the relief fronts. Now different parts of the relief fronts propagate at different speeds which, moreover, are varied in time. Later, the relief waves reach the opposite parts of the jet free surface and are reflected back into the jet as compression waves. This process will then be repeated many times, and so the intensities of the relief and compression waves vanish in time. We expect that such processes will not put any new limitations of the applicability of the acoustic approach.

Crocco' vorticity law shows that the curved parts of a shock are responsible for the generation of vorticity behind the shock front. The vorticity ω is defined here as $\omega = \partial u/\partial z - \partial v/\partial x$, where u, v are the horizontal and vertical components of the liquid velocity, respectively. A general expression for the vorticity jump on an unsteady shock front was given by Piskareva & Shugaev (1977). In our case their results yield

$$\omega_s = \frac{\rho_2}{\rho_0}\left(1-\frac{\rho_0}{\rho_2}\right)^2 \frac{\partial D_n}{\partial s}, \qquad \text{(A 12)}$$

where ω_s is the vorticity on the rear side of the shock, ρ_2 is the liquid density behind the shock front, D_n is the speed of the shock front with respect to the fluid ahead of it, and $\partial/\partial s$ is the derivative along the front. For a stationary shock (A 12) is the same as the relation found by Truesdell (1952). Taking into account the conditions at the shock front, we can rewrite (A 12) in the form

$$\omega_s = N(\delta,\gamma)\, c_0 \frac{\partial \delta}{\partial s}, \quad N(\delta,\gamma) = \frac{\surd 2(\gamma+1)\,\delta^2}{[(2+\gamma\delta)^2-\delta^2][2+(\gamma+1)\,\delta]^{1/2}}, \qquad \text{(A 13)}$$

where $\delta = \delta(s,t)$ is the shock strength on the curved parts AB and $A'B'$. We assume that $\delta(s,t)$ decreases monotonically from Δ at point A to δ_f, $0 \leqslant \delta_f \leqslant \Delta$, at point B. The exact value of δ_f is not necessary to obtain an estimation of ω_s as $M \to 0$. Near point B, the flow is of Prandtl–Meyer type and, therefore, high gradients of both the pressure and the velocity are expected here. We get $\delta = O(\Delta) = O(M)$, $s = O(M^{1/2}L)$, and $\partial\delta/\partial s = O(M^{1/2}/L)$. This predicts that $\omega_s = O(M^{3/2}V/L)$ as $M \to 0$. Thus, the flow behind the shock generated under the jet impact may be approximately considered as isentropic and irrotational when $M \ll 1$. A weak vortex wake is localized near the free surface of the jet, its intensity being of $O(M^{3/2}V/L)$ as $M \to 0$. Generally speaking, this analysis is not complete because vorticity may be generated not only by curved shocks, but by the relief waves as well. This follows from the theory of weak discontinuities. We do not know any results in this field, and this problem is not considered here.

Within the framework of the acoustic approximation the equations of motion and the boundary conditions are linear. In order to justify that at the stage under consideration the equations can be linearized, consider the Euler equations of an isentropic flow

$$v_t + M(v \cdot \nabla) v = -\frac{1}{\rho} \nabla p, \tag{A 14}$$

$$p_t + M(v \cdot \nabla p) + (1 + M\gamma p) \nabla \cdot v = 0, \tag{A 15}$$

and the equation of state

$$\rho = (1 + M\gamma p)^{1/\gamma} \tag{A 16}$$

which follows from (A 4). The equations are written in the non-dimensional variables which were introduced in §1. Here $v = (u/V, v/V)$ is the vector of the liquid velocity, and $p = (P - p_0)/(\rho_0 c_0 V)$ is the deviation of the pressure from its initial value. The assumption which leads to the acoustic approximation may be formulated as follows: the unknown functions p, v and their first derivatives remain bounded as $M \to 0$ almost everywhere in the flow region. This assumption allows us to neglect the nonlinear terms in (A 14), (A 15) and to change (A 16) for $\rho = 1 + Mp + \dots$ in the leading order. The words 'almost everywhere' are very important: they point out that the acoustic theory is expected to be valid everywhere in the flow region except for some zones, the dimensions of which are small and tend to zero as $M \to 0$. One of these zones was distinguished above: the vicinity of the point (in the three-dimensional problem of the line) where the shock front touches the jet free surface.

In order to find the liquid flow within the framework of the acoustic approximation, the deformation of the jet free surface is disregarded. We assume that the velocity of the liquid cannot be much higher than the impact speed V. Then the normal velocity of the free surface is of $O(V)$, and its deformation is of $O(Vt)$. Therefore, the surface deformation is much less than the dimension of the jet cross-section. Indeed, $O(Vt/L) = O((tc_0/L)(V/c_0)) = O(M)$ because $tc_0/L = O(1)$ at the stage under consideration.

It follows from physical reasoning that the acoustic approach fails: (i) near the fronts of the relief waves and the compression waves where the normal derivatives of the pressure and the liquid velocity change abruptly in the general case; (ii) near the contact line of the free surface with the rigid boundary, where the horizontal component of the velocity is expected to be much higher than that inside the jet. The flow structure inside each of these zones is more complex than that predicted by the acoustic theory. A powerful tool providing a complete description of the flow is given by the asymptotic methods. The main idea of the methods is based on the assumption that the flows inside the narrow zones depend on the flow inside the jet (where the acoustic approximation is valid), but the inverse influence is weak and can be approximately disregarded. Preliminary analysis indicates that the flow near the curved parts of the shock front is governed by transonic theory, and near the fronts of weak discontinuities (fronts of the relief and compression waves) by nonlinear geometric acoustic theory. It is expected that the acoustic effects may be disregarded near the contact line and an analysis similar to that given by King & Needham (1994) can be used to improve the flow description here.

The study of the fine structure of the flow inside the jet is very important because it gives us an understanding of the processes occurring under the impact. On the other hand, we expect that details of the flow in the above narrow zones make small contributions to the global characteristics. For example, the analysis of the flow near the contact line can give us an estimate of the length of the spray jet and helpful details of the liquid motion. But the pressure inside this vicinity is near atmospheric, p_0, (see Frankel 1990) and it is hard to believe that the details of the pressure distribution in the spray jet will make a significant contribution to the hydrodynamic force on the impacted surface.

Appendix B. Equation (14)

We use the standard integral (Gradshteyn & Ryzhik 1980)

$$\int_0^{\pi/2} J_0(2z\cos x)\,\mathrm{d}x = \tfrac{1}{2}\pi J_0^2(z)$$

with the help of which the left-hand side of (14) can be rewritten as

$$\int_0^t J_0^2[c(t^2-z^2)^{1/2}]\,\mathrm{d}z = \frac{2}{\pi}\int_0^{\pi/2}\int_0^t J_0[2c(t^2-z^2)^{1/2}\cos x]\,\mathrm{d}z\,\mathrm{d}x.$$

Let $2ct\cos x = b$ and introduce the new integration variable $zb/t = u$. Then the inner integral is

$$\frac{t}{b}\int_0^b J_0[(b^2-u^2)^{1/2}]\,\mathrm{d}u.$$

Here

$$\int_0^b J_0[(b^2-u^2)^{1/2}]\,\mathrm{d}u = \sin b$$

(see Gradshteyn & Ryzhik 1980). Therefore

$$\int_0^t J_0^2[c(t^2-z^2)^{1/2}]\,\mathrm{d}z = \frac{1}{\pi c}\int_0^{\pi/2}\frac{\sin(2ct\cos x)\,\mathrm{d}x}{\cos x}.$$

The final substitution $\cos x = v$ gives

$$\int_0^{\pi/2}\frac{\sin(2ct\cos x)\,\mathrm{d}x}{\cos x} = \int_0^1\frac{\sin(2ctv)\,\mathrm{d}v}{v(1-v^2)^{1/2}}.$$

Let us now transform the right-hand side of (14) using the integral representation of the Bessel function of zero order

$$J_0(z) = \frac{2}{\pi}\int_0^1\frac{\cos zv}{(1-v^2)^{1/2}}\,\mathrm{d}v.$$

We obtain

$$\int_0^{2ct} J_0(\tau)\,\mathrm{d}\tau = \frac{2}{\pi}\int_0^1\int_0^{2ct}\cos\tau v\,\mathrm{d}\tau\frac{\mathrm{d}v}{(1-v^2)^{1/2}} = \frac{2}{\pi}\int_0^1\frac{\sin(2ctv)\,\mathrm{d}v}{v(1-v^2)^{1/2}};$$

this finishes the proof of (14).

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